Alternative Connection Radius for Asymptotic Optimality in RRT*

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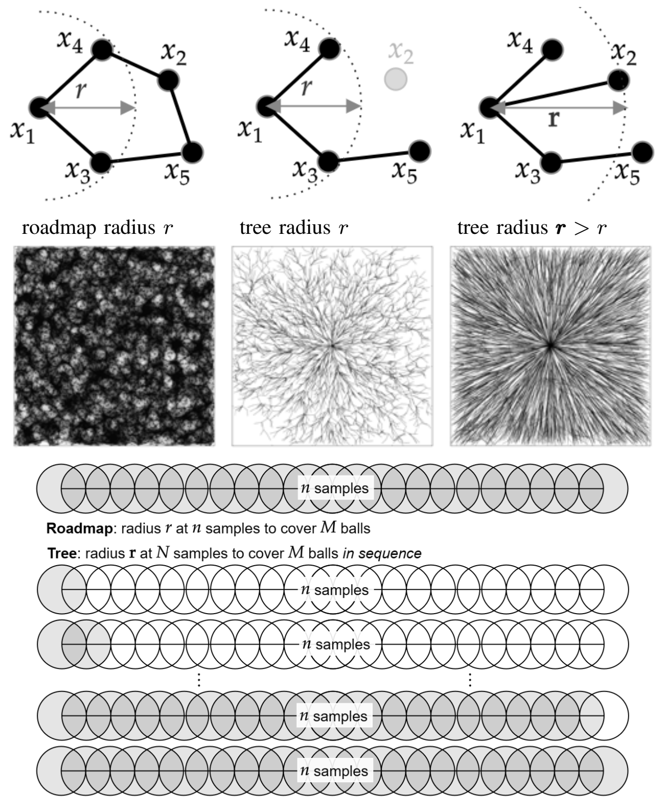
***Abstract*— Connection radius in asymptotically optimal motion planning algorithms is of interest to both understand the theoretical properties of these algorithms, as well as to ensure practical performance by estimating lower bounds. The smaller the connection radius, the sparser the data structures constructed using them, which makes the associated algorithms computationally more efficient. The original radii for both roadmap and tree variants were reported to be asymptotically shrinking functions of n. A recent amendment to the original arguments for trees demonstrated that the radius has to be larger for tree-based variants (RRT*). A practical problem in the newly proposed radius is the persistence of hard-to-estimate or large-valued parameters (like optimal path cost) within the connection radius function. In this short paper, a new perspective is presented of approaching the proof of asymptotic optimality of RRT* from a minimal variant of RRT* that only includes tree additions within connection neighborhoods. The work provides an alternative connection radius that gets rid of unwieldy parameters, presents insights that holds promise in studying the problem and using the result.**

I. Motivation and Key Findings

Asymptotic optimality [1], [2] in motion planning [3] demonstrated that in kinematic motion planning, careful selection of connection rules in roadmaps [4] and trees [5] can ensure that the cost of the solution path asymptotically approaches the optimal solution cost. The connection radius lower bound for roadmaps was subsequently improved [6]. The initial key results [3], [6] leverage theories [7] developed in random geometric graphs [8]. These theories describe probabilistic and asymptotic properties of graphs with sampled vertices that include edges for vertices lying within neighborhoods defined by a connection radius function. Careful choice of connection radius functions can guarantee asymptotic optimality of motion plans discovered over the graphs or trees. It is of significance to identify accurate lower bound estimates of these connection radii. Smaller connection radii are more desirable as they express sparser data structures and are incur lower computation and memory overheads. It is of practical interest to identify estimates for connection radii in such asymptotically motion planners. *This work provides a new connection radius lower bound for* RRT* *that is smaller and only involves known constants (Fig 1 bottom).*

The arguments applicable to PRM* [3], [6] defined a connection radius as an asymptotically diminishing function r_n of the number of samples n. The lower bound was reported for both PRM* and RRT* (i.e., both roadmap and tree-based



$$\boldsymbol{r} > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}} \left(\frac{\text{vol(free C-Space)}}{\text{vol(unit hyperball)}}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}$$

Fig. 1. ***(Top row:)*** Five states $x_1 \cdots x_5$ sampled in sequence. x_2 will be added to a roadmap (left) but is not part of the tree (middle), unless a larger radius is used (right). ***(Middle row:)*** 10000 samples connected in an 2D unit square. Using roadmap connectivity with radius proportional to $(\log n/n)^{1/d}$ (left), a radius proportional to $(\log n/n)^{1/d}$ used to grow a tree from the center (middle), and tree using a larger radius (right). The middle does not resemble an optimal tree while the right does (as has been theorized in earlier work [9]). ***(Bottom Row:)*** In this work we build on roadmap arguments to come up with an estimate for connection radius that emulates the behavior like the middle right. A sketch of the relationship between roadmap arguments of sampling M balls along an optimal path compared to tree arguments that need to guarantee the M balls in sequence using the connection radius outlined by the connection radius estimate.

versions of the approach) as proportional to $(\log n/n)^{\frac{1}{d}}$, where d is the dimensionality of the configuration space.

Recently [9], it has been shown that this radius is not sufficient to address tree-based methods like RRT*. Instead, the lower bound is shown to be *larger* and proportional to $(\log n/n)^{\frac{1}{d+1}}$, which is larger than what is necessary for roadmaps. It is important to provide an sketch of the reasoning behind this incongruence. In roadmaps, a new vertex is connected to another roadmap vertex as long as it is within the connection radius. In trees, connecting a new vertex to a tree vertex when it is within the connection radius

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and the tree vertex has already been added through sampling. Previous work [9] noted this as having to contend with the additional dimension of time (order of samples as outlined in Fig 1). The lower bound of the connection radius obtained for RRT* was discovered to be proportional to $(\log n/n)^{\frac{1}{d+1}}$. The value of this functional estimate of the connection radius is important because these estimates are necessary when RRT* is used by practitioners. A key bottleneck with the existing result is the lingering existence of a parameters in the constant of proportionality. This includes a term c^*/θ were c^* is the optimal path cost (which can be arbitrarily large) and $\theta \in (0, 1/4)$ (which can be arbitrarily small). We present both a alternative proof as well as a radius that conveniently drops these terms, albeit being slightly larger.

A. Key Results

In this short work we present the following contributions.

- We present a novel proof of asymptotic optimality of RRT* by building off a minimal AO version of the RRT* algorithm (Algo 2).
- Our new proof techniques allow us to estimate a new connection radius estimate for RRT*

$$\boldsymbol{r}_N > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}}\frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}},$$

  where N is the number of samples, d is the dimensionality of the configuration space, μ_{free} is the volume of the feasible configuration space, and λ is the volume of an unit hyperball. This connection radius is free from a c^*/θ term that can be hard to estimate, though asymptotically our radius is still larger than the state-of-the-art.

II. Roadmap Preliminaries

A robot r with d degrees of freedom is represented as a d-dimensional configuration $x \in \mathcal{X} \subset \mathbb{R}^d$. The robot is situated alongside obstacles in a workspace $\mathcal{W} \subset \mathbb{R}^3$. The subset of the workspace occupied by the obstacles is $\mathcal{W}_{\text{obs}} \subset \mathcal{W}$. The robot geometries themselves occupy $vol(x) \subset \mathcal{W}$ that for the configuration x. The invalid subset of the configuration space is defined as $\mathcal{X}_{\text{obs}} = \{x \;\; s.t. \; vol(x) \cap \mathcal{W}_{\text{obs}} \neq \emptyset, \; x \in \mathcal{X}\}$. The feasible subset is $\mathcal{X}_{\text{free}} = \mathcal{X} \setminus \mathcal{X}_{\text{obs}}$.

A motion planning problem is defined by a start configuration $x_s \in \mathcal{X}$, a goal state $x_g \in \mathcal{X}$. The solution to the motion planning problem is a feasible path $\pi : \mathcal{X}_{\text{free}} \rightarrow [0, 1], \; \pi(0) = x_s, \pi(1) = x_g$. Some cost function is defined for $c : \pi \rightarrow \mathbb{R}^{\geq 0}$. Given all possible feasible trajectories Π, the optimal motion planning problem is then solved when the optimal solution π^* is found such that $\pi^* \in \underset{\pi \in \Pi}{\texttt{argmin}}\; c(\pi)$.

Definition 1 (Asymptotic Optimality (AO) in Path Cost): In sampling-based algorithms, as the number of samples n increases, asymptotic optimality ensures that the solution π_n approaches[1] arbitrarily close to the optimal cost, $\lim_{n\to\infty} \Pr(\{c(\pi_n) \leq (1+\epsilon)c^*\}) \rightarrow 1, \;\; \forall \epsilon > 0$.

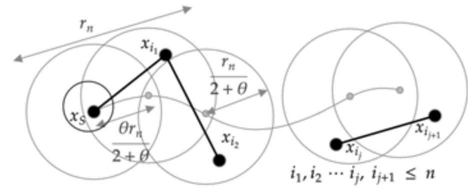

Fig. 2. Sampling events are studied over an asymptotic construction along a robustly optimal trajectory. Here a sequence of hyperballs of radius $\frac{r_n}{2+\theta}$ and separated by $\frac{\theta r_n}{2+\theta}$ are described. Samples x_{1_j} are associated with indices $\imath_j$ indicating when the configuration was sampled. In a roadmap, the samples are considered in useful to hit hyperballs in any order, regardless of their index (when they were sampled).

Algorithm 1: AOROADMAP [3], [6]

Input: Number of configuration samples n, connection radius r_n, start x_s, goal x_g
Output: Path π_n
$\mathcal{G}_n(\mathcal{V}_n, \mathcal{E}_n) \leftarrow (\emptyset, \emptyset)$;
for n *times* **do**
  $x \leftarrow$UNIFORMSAMPLE($\mathcal{X}$);
  $\mathcal{N} \leftarrow$NN($x, \mathcal{V}_n, r_n$);
  for $x_{\text{nn}} \in \mathcal{N} \wedge$ ISFEASIBLE(x_{nn}, x) **do**
    $\mathcal{V}_n \leftarrow \mathcal{V}_n \cup x$;
    $\mathcal{E}_n \leftarrow \mathcal{E}_n \cup (x_{\text{nn}}, x)$;
return $\pi_n \leftarrow$A*($\mathcal{G}_n, x_s, x_g$);

Theorem 1 (AO of roadmaps [3], [6] in Algo 1): A roadmap constructed with a connection radius r_n provides asymptotic guarantees of discovering a solution π_n after n samples that is arbitrarily close in cost to c^* of a robustly optimal path π^*, when $r_n = \frac{\gamma_{\text{rm}}(\log n)^{\frac{1}{d}}}{n^{\frac{1}{d}}}, \;\; \gamma_{\text{rm}} > 2\left(\frac{1}{d}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}}$.

A. Elements of Roadmap Proof: Construction

An essential component of the proof of asymptotic optimality for roadmaps is an asymptotic construction (Fig 2). Each hyperball is of size (radius) $\frac{r_n}{2+\theta}$, separated by $\frac{\theta r_n}{2+\theta}$ for some arbitrarily small $\theta \in (0, 1)$. The length of the optimal path is c^*, then $M_n = \frac{c^*(2+\theta)}{\theta r_n}$. Assign a $\mathcal{B}_{\frac{r_n}{2+\theta}}(\cdot)$ ball around configurations $(\bar{x}_i)$ along π^* and number them as $\mathcal{B}_{\frac{r_n}{2+\theta}}(\bar{x}_1), \mathcal{B}_{\frac{r_n}{2+\theta}}(\bar{x}_2) \cdots \mathcal{B}_{\frac{r_n}{2+\theta}}(\bar{x}_{M_n})$. We are interested in the probability event of hitting all the balls. Tracking the construction of M_n hyperballs of radius $\frac{r}{2+\theta}$ along π^* we get the following probability bound for k samples.

$$\Pr(\{\text{Fail to sample ball after } k \text{ tries}\}) \tag{1}$$

$$\leq \left(1 - \frac{\lambda r^d}{\mu_{\text{free}}(2+\theta)^d}\right)^k \leq \exp\left(\frac{-\lambda k r^d}{\mu_{\text{free}}(2+\theta)^d}\right) \tag{2}$$

The first inequality tracks the failure event of sampling a ball of volume $\lambda(\frac{r_n}{2+\theta})^d$. The second inequality follows from applying the union bound over the M_n events. This is asymptotically assured to not happen in a roadmap for $r = r_n, \; k = n$ as described in Thm 1.

[1] Note that robust optimality has been very carefully described in previous work [6], [9] which needs π_n to *approach* some robust π_ϵ that is arbitrarily close (ϵ close in cost and δ_ϵ-clear) to the optimal π^*. For clarity, at the risk of some imprecision, in this manuscript we mean robustly (near-)optimal trajectories wherever sampling arguments are applied.

Algorithm 2: AOTREE (CURRENT WORK)

```
Input: Number of configuration samples N, connection
       radius r_N, start x_s, goal region X_goal
Output: Path π_N
T_N(V_N, E_N) ← (x_s, ∅);
for N times do
    x ← UNIFORMSAMPLE(X);
    N ← NN(x, V_N, r_N);
    x_best ← argmin_{x_nn ∈ N ∧ ISFEASIBLE(x_nn, x)} COSTTOGO(x_nn) + c(x_nn, x)
    V_N ← V_N ∪ x;
    E_N ← E_N ∪ (x_nn, x);
return π_N ← RETRACEPATH(T_N, X_goal);
```

III. NEW ARGUMENTS FOR TREES

This section will introduce the core contribution of this paper, a new set of theoretical arguments to derive a new estimate of a lower bound for the connection radius necessary for preserving asymptotic optimality in RRT*. Towards this end, we will first introduce Algo 2 and then RRT* as Algo 3. The highlighted lines indicate a) the modifications of Algo 2 wrt Algo 1 and b) the modifications of Algo 3 wrt Algo 2.

Theorem 2 (If Algo 2 is AO, Algo 3 is AO.): Considering $\boldsymbol{r}_N < r_n(n)$, where $\boldsymbol{r}_N$ is the connection radius used through the entirety of Algo 2, and $r_n(n)$ is the smallest radius used in the final iteration of Algo 3, if Algo 2 is asymptotically optimal, so is Algo 3.

Proof: When $\boldsymbol{r}_N < r_n(n)$, the connection radius used in Algo 2 is no greater than the radius used in Algo 3. Algo 2 is designed to only add edges connected to tree vertices within the connection radius. This is in contrast to RRT* where additional steering edges (Algo 3 lines 6-9) add edges to the tree for samples Algo 2 will not. This means that Algo 3 (lines 6-9) introduces a strictly larger set of edges to the tree. This proves that, if Algo 2 is AO, additional edges will preserve the property. Another alteration in RRT* is the rewiring in lines 13-15. Given the construction in Fig 3, if Algo 2 traces such a construction, edges being added will asymptotically be optimal into each hyperball. This means the condition on Algo 3 line 14 will not trigger. ■

We can, hereon, focus on AO arguments for Algo 2 and let the results follow to RRT*. Notably, Algo 2 is a modification on top of Algo 1 which, instead of adding all the neighbors, will only add the best neighbor out of the tree vertices. We will use r_n as the roadmap connection radius and the new one of interest for trees to be $\boldsymbol{r}_N$.

Theorem 3 (Algo 2 is AO): Algo 2 is asymptotically optimal when $\boldsymbol{r}_N > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}}\frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}$.

A. Proof of Probabilistic Events

The following arguments will build off the asymptotic guarantees that are set up by the asymptotically optimal roadmap across its construction (Fig 3.

A tree would require the balls to be hit in sequence, so

$$\mathcal{B}(\bar{x}_1) \textbf{ then } \mathcal{B}(\bar{x}_2)\cdots \textbf{ then } \mathcal{B}(\bar{x}_{M_n})$$

Algorithm 3: RRT* [3], [9]

```
   Input: Number of configuration samples n, connection
          radius r_n, start x_s, goal region X_goal
   Output: Path π_n
1  T_n(V_n, E_n) ← (x_s, ∅);
2  for n times do
3      x ← UNIFORMSAMPLE(X);
4      r ← r_n(|V_n|);
5      N ← NN(x, V_n, r);
6      if N = ∅ ∧ η > r then
7          x_tree ← NEAREST(x, V_n);
8          x ← STEER(x_tree, x, η);
9          N ← NN(x, V_n, r);
10     x_best ← argmin_{x_nn ∈ N ∧ ISFEASIBLE(x_nn, x)} COSTTOGO(x_nn) + c(x_nn, x)
11     V_n ← V_n ∪ x;
12     E_n ← E_n ∪ (x_nn, x);
13     for x_nn ∈ N do
14         if ISFEASIBLE(x_nn, x) ∧ COSTTOGO(x) +
             c(x, x_nn) < COSTTOGO(x_nn) then
15             REWIRE(x_nn, x);
16 return π_n ← RETRACEPATH(T_n, X_goal);
```

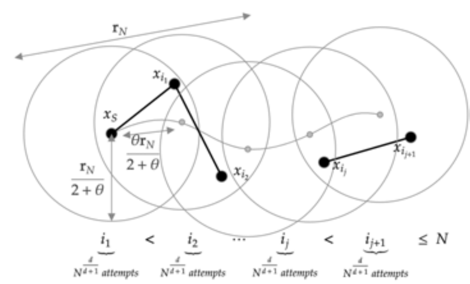

Fig. 3. The construction is identical to one described fro roadmaps. The connection radius $\boldsymbol{r}_N$ required is larger for trees. The sampling indices are important for tree arguments because they need to be sampled in sequence $i_1 < i_2 \cdots i_j < i_{j+1}$ for consecutive hyperballs up to the last M_n ball.

We want to use a new functional estimate for the connection radius $\boldsymbol{r}_N$ necessary for trees using N samples as $\boldsymbol{r}_N = \frac{\gamma_{\text{ours}}(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}$.

Intuition behind Estimate: We can think of a new functional estimate for the connection radius as building on the necessity to allocate the asymptotic probability guarantees to each of the indexed hyperballs in sequence, analogous to allocating a *bucket of samples* to each hyperball event in sequence. We already demonstrated n samples for the M_n ball construction works (for a roadmap). If we repeat this for each ball, roughly, we need n samples for each of the M_n balls, i.e., the new radius $\boldsymbol{r}$ as a function of tree samples N needs $N \geq nM_n \implies n < N^{\frac{d}{d+1}}$. The estimate of the tree radius at N can be at least the roadmap radius at $r(n)$, then.

$$\boldsymbol{r}(N) > r(N^{\frac{d}{d+1}}) > \frac{\gamma_{\text{rm}}(\log N^{\frac{d}{d+1}})^{\frac{1}{d}}}{(N^{\frac{d}{d+1}})^{\frac{1}{d}}} \geq \frac{\gamma_{\text{ours}}(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}$$

New Construction for Trees: We will track the number of tree samples as N, the connection radius for trees using our functional estimate as $\boldsymbol{r}_N$. For each (i^{th}) ball $\mathcal{B}(\bar{x}_i)$ we will allocate $k = N^{\frac{d}{d+1}}$ samples. Using the same θ parameterized separation of balls, we get the number of hyperballs as $M_N = \frac{c^*(2+\theta)}{\theta\boldsymbol{r}_N}$, each with radius $\frac{\boldsymbol{r}_N}{2+\theta}$. *We want to repeat the probabilistic event outlined in (Eqs 2) over the construction*

M_N times giving at least $N^{\frac{d}{d+1}}$ attempts for ***each ball*** *in the construction.* This is possible since $N > M_N N^{\frac{d}{d+1}}$. Here we need to show $\boldsymbol{r}_N$ demonstrates asymptotic guarantees in terms of hitting the probabilistic event of sampling across each of the $i = 1 \cdots M_N$ hyperballs ($\mathcal{B}_{\frac{\boldsymbol{r}_N}{2+\theta}}(\bar{x}_i)$) construction using $N^{\frac{d}{d+1}}$ samples.

$$
\begin{aligned}
&\Pr(\{\text{Fail to sample } \mathcal{B}_{\frac{\boldsymbol{r}_N}{2+\theta}}(\bar{x}_i) \text{ after } N^{\frac{d}{d+1}} \text{ attempts}\}) \\
&\leq \exp\left(\frac{-\lambda}{\mu_{\text{free}}(2+\theta)^d} N^{\frac{d}{d+1}} \boldsymbol{r}_N^d\right) \\
&\leq \exp\left(\frac{-\lambda N^{\frac{d}{d+1}} \gamma_{\text{ours}}^d}{\mu_{\text{free}}(2+\theta)^d} \left(\frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}\right)^d\right) \\
&\leq \exp\left(\frac{-\lambda \gamma_{\text{ours}}^d}{\mu_{\text{free}}(2+\theta)^d} \log N\right) \\
&\Pr(\{\text{Fail to sample } M_N \text{ construction}\}) \\
&\leq \cup_i^{M_N} \Pr(\{\text{Fail to sample } \mathcal{B}_{\frac{\boldsymbol{r}_N}{2+\theta}}(\bar{x}_i) \text{ between attempts} \\
&\qquad\qquad (i-1)N^{\frac{d}{d+1}} \text{ and } iN^{\frac{d}{d+1}}\}) \\
&\leq M_N \exp\left(\frac{-\lambda \gamma_{\text{ours}}^d}{\mu_{\text{free}}(2+\theta)^d} \log N\right) \\
&\leq \frac{c^*(2+\theta)}{\theta(\log N)^{\frac{1}{d}}} N^{\frac{1}{d+1}} N^{\frac{-\lambda \gamma_{\text{ours}}^d}{\mu_{\text{free}}(2+\theta)^d}}
\end{aligned}
$$

For asymptotic convergence we need the exponent of N to be negative so that the probability goes to 0.

$$
\begin{aligned}
&\lim_{n\to\infty} \Pr(\{\text{Fail to sample } M_N \text{ constructions}\}) \to 0 \\
&\implies \gamma_{\text{ours}}^d > \frac{1}{d+1} \frac{\mu_{\text{free}}(2+\theta)^d}{\lambda} \\
&\implies \gamma_{\text{ours}} > (2+\theta)\left(\frac{1}{d+1}\right)^{\frac{1}{d}} \left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}}
\end{aligned}
$$

We get as the connection radius,

$$
\boxed{\boldsymbol{r}_N > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}} \left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}} \tag{3}
$$

for arbitrarily small θ which is a proof parameter.

By tracing the indicator events along smaller hyperballs $\beta \boldsymbol{r}_N$ in the construction (as in previous work [3], [6], [9] and outlined in the Appendix) we can demonstrate that the cost of the solution traced along the samples connected along the construction will be arbitrarily close to c^*. This proves that for the connection radius $\boldsymbol{r}_N$, Algo 2 is AO, proving Thm 3. By Thm 2, $\boldsymbol{r}_N$ also guarantees RRT* to be asymptotically optimal using this connection radius. ■

Note *on the connection radius $\boldsymbol{r}_N$ being asymptotically larger than the one presented in the amendment to* RRT* *[9] (due to the exponent on the $\log N$ term). Nonetheless, we arrive at this bound that drops some proof parameters that might be hard to estimate using Algo 2 as the foundation for our argument. This is of practical and theoretical utility.*

Notes on the parameter θ are included in the Appendix.

Note *on Algo 2 as a testbed for* RRT* *properties. The amendment [9] to the proof of* RRT* *[3] pointed out that the theoretical connection radius should be insufficient unless expanded. Here we show the growth of the tree in Algo 2 as the stripped-down version of* RRT* *using purely the edges that are guaranteed to contribute to the AO properties (visualized in Fig 4). Indeed, as pointed out in previous work [9], despite apparent theoretical inconsistencies,* RRT* *[3] has worked well in practice, even with the originally proposed radius. This raises interesting questions on how the additional operations in* RRT* *(*RRT*-like propagations and rewiring) might potentially alter its AO properties, possibly lowering the currently proposed bounds.*

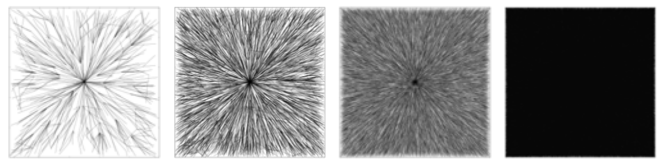

Fig. 4. The growth of a tree out of the center of an unit 2D configuration space using Algo 2. From left to right, the figures show the result after 10^3, 10^4, 10^5, and 10^6 iterations respectively.

IV. Conclusion

The currently proposed proof techniques provide a slightly different perspective compared to previous work [9], based off arguments built on a minimal version of RRT* (Algo 2) for asymptotic optimality. A benefit is the elimination of a c^*/θ term in the connection radius. The practical niceties primarily provide us a connection radius value that we can use and be assured possesses theoretical guarantees. Algo 2 provides opportunities for furthering investigation into the properties of RRT* and asymptotic optimality. The current work hopefully takes a small but positive step towards reconciling practice and theory of using RRT* for asymptotically optimal motion planning.

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APPENDIX

A. $\boldsymbol{r}_N$ guarantees β-hyperball events for optimality

These arguments are almost identical to previous work [9], being adapted to our construction. We will inspect events of hitting hyperballs shrunk by a factor of $\beta \in (0,1)$, i.e, assign an indicator $\mathbb{I}_i = 1$ for the i^{th} ball with the failure event {Fail to sample $\mathcal{B}_{\frac{\beta \boldsymbol{r}_N}{2+\theta}}(\bar{x}_i)$ after $N^{\frac{d}{d+1}}$ attempts}. The expected value is $E(\mathbb{I}_i)$ is as follows.

$$E(\mathbb{I}_i) = \Pr(\{\text{Fail to sample } \mathcal{B}_{\frac{\beta \boldsymbol{r}_N}{2+\theta}}(\bar{x}_i) \text{ after } N^{\frac{d}{d+1}} \text{ attempts}\}) \\ \leq \exp\left(\frac{-\lambda \beta^d \gamma_{\text{ours}}^d}{\mu_{\text{free}}(2+\theta)^d}\log N\right) \leq N^{\frac{-\beta^d}{d+1}}$$

Define K^β as the total count of hyperballs in our construction that successfully sample within the β-shrunk ball in the sequential allocation of $N^{\frac{d}{d+1}}$ samples. This means, $E(K^\beta) = \sum_i^{M_N} E(\mathbb{I}_i) \leq M_N E(\mathbb{I}_i)$. We can now describe an event describing the probability of the K^β being less than some fraction ($\alpha \in (0,1)$) of the total number M_N.

$$\Pr(\{K^\beta > \alpha M_N\}) \leq \frac{E(K^\beta)}{\alpha M_N} \leq \frac{N^{\frac{-\beta^d}{d+1}}}{\alpha}.$$

For arbitrarily small constant α, β

$$\lim_{N\to\infty} \Pr(\{K^\beta > \alpha M_N\}) \to 0.$$

Algo 2 guarantees that the consecutive hyperballs will be added to the tree, so the event $\{K^\beta > \alpha M_N\}$ lets us bound the cost of the discovered path in terms of pairs of hyperballs being sampled within the $\frac{\theta \boldsymbol{r}_N}{2+\theta}$ ball or $\frac{\beta\theta \boldsymbol{r}_N}{2+\theta}$ ball. For a segment involving consecutive balls with at least one β event, the cost inflation is at most $\frac{2\beta\theta \boldsymbol{r}_N}{2+\theta}$, while the cost inflation otherwise is at most $\frac{2\theta \boldsymbol{r}_N}{2+\theta}$. This means, the bound of the sum of segment costs will be

$$c(\pi_N) = \sum_i^{M_N} \text{Edge between } \mathcal{B}(\bar{x}_{i-1}) \text{ and } \mathcal{B}(\bar{x}_i) \\ \leq \alpha M_N\left(\frac{\theta \boldsymbol{r}_N}{2+\theta} + \frac{2\theta \boldsymbol{r}_N}{2+\theta}\right) + (1-\alpha)M_N\left(\frac{\theta \boldsymbol{r}_N}{2+\theta} + \frac{2\beta\theta \boldsymbol{r}_N}{2+\theta}\right) \\ \leq \frac{M_N \theta \boldsymbol{r}_N}{2+\theta} + \frac{M_N \theta \boldsymbol{r}_N}{2+\theta}(2\alpha + 2\beta) \leq c^*(1 + 2\alpha + 2\beta) \\ \implies \exists \alpha, \beta > 0, \forall \epsilon > 0,\ s.t., c(\pi_N) \leq (1+\epsilon)c^* \quad \blacksquare$$

B. Values of Proof Parameter θ

Note that nominally, the connection radius is $\boldsymbol{r}_N = \gamma_{\text{ours}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}} > (2+\theta)\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}}$. While $\theta \in (0,1)$ we can flesh out its bound.

We need in our construction $N \geq N^{\frac{d}{d+1}} M_N$. This means, $N \geq N^{\frac{d}{d+1}} \frac{c^*(2+\theta)}{\theta \boldsymbol{r}_N}$, implies $\theta \geq \frac{c^*(2+\theta)}{\gamma_{\text{ours}}(\log N)^{\frac{1}{d}}}$. Plugging this into the radius we get

$$\boldsymbol{r}_N > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}} \\ + \theta\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}} \\ \implies \boldsymbol{r}_N > 2\left(\frac{1}{d+1}\right)^{\frac{1}{d}}\left(\frac{\mu_{\text{free}}}{\lambda}\right)^{\frac{1}{d}} \frac{(\log N)^{\frac{1}{d}}}{N^{\frac{1}{d+1}}} + \frac{c^*}{N^{\frac{1}{d+1}}}$$

Hence, regardless of the choice of the proof parameter, for large enough N, Eq 3 should suffice.